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ORIGINAL PAPER

A note on the principal measure and distributional (*p, q*)-chaos of a coupled lattice system related with Belusov–Zhabotinskii reaction

Risong Li · Xiaoliang Zhou · Yu Zhao · Cunyou Huang

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Abstract García Guirao and Lampart in (J Math Chem 48:159–164, 2010) presented a lattice dynamical system stated by Kaneko in (Phys Rev Lett 65:1391–1394, 1990) which is related to the Belusov–Zhabotinskii reaction. In this paper, we prove that for any non-zero coupling constant $\varepsilon \in (0, 1)$, this coupled map lattice system is distributionally (p, q)-chaotic for any pair $0 \leq p \leq q \leq 1$, and that its principal measure is not less than $(1-\varepsilon)\mu_p(f)$. Consequently, the principal measure of this system is not less than

$$(1-\varepsilon)\left(\frac{2}{3}+\sum_{n=2}^{\infty}\frac{1}{n}\frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}\right)$$

for any non-zero coupling constant $\varepsilon \in (0, 1)$ and the tent map Λ defined by

$$\Lambda(x) = 1 - |1-2x|, \quad x \in [0, 1].$$

R. Li (✉) · X. Zhou · Y. Zhao · C. Huang
School of Science, Guangdong Ocean University, Zhanjiang 524025, People's Republic of China
e-mail: gdoulrs@163.com

X. Zhou
e-mail: zjhdzxl@yahoo.com.cn

Y. Zhao
e-mail: datom@189.cn

C. Huang
e-mail: hdhcy1962@21cn.com

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1 Introduction

By a topological dynamical system (t.d.s. for short) (X, f) we mean a compact metric space X together with a continuous map $f : X \to X$. Since Li and Yorke [1] introduced the term of chaos in 1975, topological dynamical systems were highly considered and studied in the literature (see [2,3]) because they are very good examples of problems coming from the theory of topological dynamics and model many phenomena from biology, physics, chemistry, engineering and social sciences.

Coming from physical/chemical engineering applications, such as digital filtering, imaging and spatial vibrations of the elements which compose a given chemical product, a generalization of classical discrete dynamical systems has recently appeared as an important subject for investigation, we mean the so called Lattice Dynamical Systems or 1d Spatiotemporal Discrete Systems. In [4] one can see the importance of these type of systems.

To analyze when one of these type of systems has a complicated dynamics or not by the observation of one topological dynamical property is an open problem (see [5]). In [5], by using the notion of chaos, the authors characterized the dynamical complexity of a coupled lattice system stated by Kaneko in [6] (for more details see for references therein) which is related to the Belusov–Zhabotinskii reaction. They proved that this coupled map lattice (CML) system is chaotic in the sense of both Devaney and Li–Yorke for zero coupling constant. Also, some problems on the dynamics of this system were stated by them for the case of having non-zero coupling constants. Recently, in [7] the authors proved that for any non-zero coupling constant $\varepsilon \in (0, 1)$, this system is chaotic in the sense of both Li–Yorke and positive entropy.

The notion of distributional chaos, introduced by Schweizer and Smítal in [8], is very interesting and important, mainly because it is equivalent to positive topological entropy and some other concepts of chaos when restricted to the compact interval case [8] or hyperbolic symbolic spaces [9]. It is also remarkable that this equivalence does not transfer to higher dimensions, e.g. positive topological entropy does not imply distributional chaos in the case of triangular maps of the unit square [10] (the same happens when the dimension is zero [11]). In [12] the authors gave a distributional chaotic minimal system.

More recently, in [13] they proved that for any non-zero coupling constant $\varepsilon \in (0, 1)$ and any pair $0 \le p \le q \le 1$, the following coupled lattice system is distributionally (p, q)-chaotic:

$$x_n^{m+1} = (1-\varepsilon) f(x_n^m) + \frac{1}{2}\varepsilon \left[f(x_{n-1}^m) + f(x_{n+1}^m) \right], \tag{1}$$

where m is discrete time index, n is lattice side index with system size L (i.e., $n = 1, 2, \ldots, L$), $\varepsilon \in (0, 1)$ is coupling constant and $f(x)$ is the unimodal map on I (i.e., $f(0) = f(1) = 0$ and f has unique critical point c with $0 < c < 1$ and $f(c) = 1$.) They also showed that for any non-zero coupling constant $\varepsilon \in (0, 1)$, the principal measure of this system is not less than $\frac{2}{3}+\sum_{n=2}^{\infty}\frac{1}{n}\frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}$ where the tent map Λ defined by $\Lambda(x)=1-|1-2x|$, $x\in[0,1]$. Inspired by these results, we will further investigate into the dynamical properties of the following lattice dynamical system:

$$x_n^{m+1}=(1-\varepsilon)f(x_n^m)+\frac{1}{2}\varepsilon\left[f(x_{n-1}^m)-f(x_{n+1}^m)\right], \tag{2}$$

where m is discrete time index, n is lattice side index with system size L, $\varepsilon\in(0,1)$ is coupling constant and $f(x)$ is the unimodal map on I. In particular, we prove that for any non-zero coupling constant $\varepsilon\in(0,1)$ and any pair $0\le p\le q\le 1$, this CML system is distributionally (p,q)-chaotic, and that its principal measure is not less than $(1-\varepsilon)\mu_p(f)$. Consequently, the principal measure of this system is not less than $(1-\varepsilon)\left(\frac{2}{3}+\sum_{n=2}^{\infty}\frac{1}{n}\frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}\right)$ for any non-zero coupling constant $\varepsilon\in(0,1)$ and the tent map Λ.

2 Preliminaries

Firstly we recall some notations and some concepts. Throughout this paper, X is a compact metric space with metric d, (X,f) is a topological dynamical system (t.d.s. for short) and $I=[0,1]$.

A pair of points $x,y\in X$ is called a Li–Yorke pair of system (X,f) if the following conditions are satisfied:

(1) $\limsup_{n\to\infty} d(f^n(x),f^n(y))>0$.
(2) $\liminf_{n\to\infty} d(f^n(x),f^n(y))=0$.

A subset $S\subset X$ is called a LY-scrambled set for f (Li–Yorke set) if the set S has at least two points and every pair of distinct points in S is a Li–Yorke pair. A system (X,f) or a map $f:X\to X$ is said to be chaotic in the sense of Li–Yorke if it has an uncountable scrambled set.

Let (X,f) be a t.d.s.. For any pair of points $x,y\in X$ and for a given $n\in\mathbb{N}$, the distributional function $F_{xy}^n:\mathbb{R}^+\to[0,1]$ is defined by

$$F_{xy}^n(t)=\frac{1}{n}\sharp\{i\in\mathbb{N}: d(f^i(x),f^i(y))<t,\ 1\le i\le n\},$$

where $\mathbb{R}^+=[0,+\infty)$ and $\sharp$ means the cardinality. Let

$$F_{xy}(t,f)=\liminf_{n\to\infty}F_{xy}^n(t)$$

and

$$F_{xy}^*(t,f)=\limsup_{n\to\infty}F_{xy}^n(t).$$

Given $0 \le p \le q \le 1$, a t.d.s. (X, f) is distributionally (p, q)-chaotic if there exist an uncountable subset $S \subset X$ and $\varepsilon > 0$ such that $F_{xy}(t, f) = p$ and $F^*_{xy}(t, f) = q$ for any pair of distinct points $x, y \in S$ and any $t \in (0, \varepsilon)$. Particularly, (X, f) is distributionally chaotic if it is distributionally (0, 1)-chaotic (see [13,14]).

The principal measure $\mu_p(f)$ of a t.d.s. (X, f) is defined by

$$\mu_p(f) = \sup_{x,y\in X} \frac{1}{D} \int_0^{+\infty} \Big(F^*_{xy}(t, f) - F_{xy}(t, f) \Big)\, dt$$

where D =diam (X) is the diameter of the space X (see [15]). It is known from [15] that

$$\mu_p(\Lambda) = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n + 1)(2^{n-1} + 1)}.$$

The state space of LDS (Lattice Dynamical System) is the set

$$\mathcal{X} = \{x : x = \{x_i\}, x_i \in \mathbb{R}^a, i \in \mathbb{Z}^b, \|x_i\| < \infty\}.$$

where $a \ge 1$ is the dimension of the range space of the map of state x_i, $b \ge 1$ is the dimension of the lattice and the l^2 norm

$$\|x\|_2 = \left(\sum_{i\in\mathbb{Z}^b} |x_i|^2 \right)^{\frac{1}{2}}$$

is usually taken ($|x_i|$ is the length of the vector x_i) (see [5]).

We will deal with the following Coupled Map Lattice system stated by Kaneko in [6] (for more details see for references therein) which is related to the Belusov–Zhabotinskii reaction (for this point we refer to [16], and for experimental study of chemical turbulence by this method one can see [17–19]):

$$x_n^{m+1} = (1 - \varepsilon) f\left(x_n^m\right) + \frac{1}{2}\varepsilon \left[f(x_{n-1}^m) - f(x_{n+1}^m) \right], \tag{3}$$

where m is discrete time index, n is lattice side index with system size L, $\varepsilon \in (0, 1)$ is coupling constant and $f(x)$ is the unimodal map on I.

In general, one of the following periodic boundary conditions of the system (3) is assumed:

(1) $x_n^m = x_{n+L}^m$,
(2) $x_n^m = x_n^{m+L}$,
(3) $x_n^m = x_{n+L}^{m+L}$,

standardly, the first case of the boundary conditions is used.

3 Main results

The system (3) was investigated by many authors, mostly experimentally or semi-analytically than analytically. The first paper with analytic results is [20], where the authors proved that this system is Li–Yorke chaotic. In [5] the authors gave an alternative and easier proof of this result.

Let d be the product metric on the product space I^L, i.e.,

$$d((x_1, x_2, \dots, x_L), (y_1, y_2, \dots, y_L)) = \left(\sum_{i=1}^{L}(x_i - y_i)^2\right)^{\frac{1}{2}}$$

for any $(x_1, x_2, \dots, x_L), (y_1, y_2, \dots, y_L) \in I^L$.

Now we define a map $F : (I^L, d) \to (I^L, d)$ by $F(x_1, x_2, \dots, x_L) = (y_1, y_2, \dots, y_L)$ where $y_i = (1-\varepsilon)f(x_i) + \frac{\varepsilon}{2}(f(x_{i-1}) - f(x_{i+1}))$. It is clear that the system (3) is equivalent to the above system (I^L, F). It is noted that the system (3) is different from the system (1) when $\varepsilon \neq 0$, and that the system (1) is equivalent to the system (I^L, F) (see [13]). In [5] the authors pointed out that for non-zero couplings constants, this lattice dynamical system (3) is more complicated.

In [13] the authors proved that the system (1) is distributionally (p, q)-chaotic for any pair $0 \le p \le q \le 1$ and any $\varepsilon \in (0, 1)$ when $f = \Lambda$. Inspired by this result and its proof we have the following result.

Theorem 3.1 *The system (3) is distributionally (p, q)-chaotic for any pair $0 \le p \le q \le 1$ and any $\varepsilon \in (0, 1)$ when $f = \Lambda$.*

Proof Fix $\varepsilon \in (0, 1)$ and a pair $0 \le p \le q \le 1$. Then, from Proposition 3 in [13] we know that the tent map Λ is distributionally (p, q)-chaotic, i.e., there exist an uncountable subset $A \subset I$ and $\delta > 0$ such that for any pair of points $x, y \in A$ with $x \neq y$ and any $0 < t < \delta$, we have

$$F_{xy}(t, \Lambda) = p \tag{4}$$

and

$$F^{xy}(t, \Lambda) = q. \tag{5}$$

Write

$$\mathcal{F} = \{(x, x, \dots, x) \in I^L : x \in A\}$$

and

$$\overrightarrow{x} = (x, x, \dots, x)$$

for any $(x, x, \dots, x) \in \mathcal{F}$. Then, for any pair of points

$$\overrightarrow{x} = (x, x, \dots, x), \quad \overrightarrow{y} = (y, y, \dots, y) \in \mathcal{F}$$

with $\overrightarrow{x} \neq \overrightarrow{y}$ and any $n \in \mathbb{N}$, we have

$$F^n(\overrightarrow{x}) = (1-\varepsilon)\overrightarrow{\Lambda^n(x)} = (1-\varepsilon)(\Lambda^n(x), \Lambda^n(x), \ldots, \Lambda^n(x)) \tag{6}$$

and

$$F^n(\overrightarrow{y}) = (1-\varepsilon)\overrightarrow{\Lambda^n(y)} = (1-\varepsilon)(\Lambda^n(y), \Lambda^n(y), \ldots, \Lambda^n(y)). \tag{7}$$

From (4), (5), (6) and (7) it follows that for any $t \in (0, \sqrt{L}\delta(1-\varepsilon))$,

$$\begin{aligned} F_{\overrightarrow{x}\overrightarrow{y}}(t, F) &= \liminf_{n\to\infty} \frac{1}{n}\sharp\left\{i \in \mathbb{N} : d(F^i(\overrightarrow{x}), F^i(\overrightarrow{y})) < t, 1 \leq i \leq n\right\} \\ &= \liminf_{n\to\infty} \frac{1}{n}\sharp\left\{i \in \mathbb{N} : |\Lambda^i(x), \Lambda^i(y)| < \frac{t}{(1-\varepsilon)\sqrt{L}}, 1 \leq i \leq n\right\} \\ &= F_{xy}\left(\frac{t}{(1-\varepsilon)\sqrt{L}}, \Lambda\right) \end{aligned} \tag{8}$$

and

$$\begin{aligned} F^{\overrightarrow{x}\overrightarrow{y}}(t, F) &= \liminf_{n\to\infty} \frac{1}{n}\sharp\left\{i \in \mathbb{N} : d(F^i(\overrightarrow{x}), F^i(\overrightarrow{y})) < t, 1 \leq i \leq n\right\} \\ &= \limsup_{n\to\infty} \frac{1}{n}\sharp\left\{i \in \mathbb{N} : |\Lambda^i(x), \Lambda^i(y)| < \frac{t}{(1-\varepsilon)\sqrt{L}}, 1 \leq i \leq n\right\} \\ &= F^{xy}\left(\frac{t}{(1-\varepsilon)\sqrt{L}}, \Lambda\right). \end{aligned} \tag{9}$$

Since A is uncountable, $\mathcal{F}$ is uncountable. Consequently, by the definition the system (3) is distributionally (p, q)-chaotic. Thus, the proof is finished. □

In [13] the authors proved that the principal measure of the system (1) is not less than $\mu_p(f)$ for any $\varepsilon \in (0, 1)$. Inspired by this result and its proof we have the following result.

Theorem 3.2 *The principal measure of the system (3) is not less than* $(1-\varepsilon)\mu_p(f)$ *for any* $\varepsilon \in (0, 1)$.

Proof For any pair of distinct points

$$\overrightarrow{x} = (x, x, \ldots, x), \quad \overrightarrow{y} = (y, y, \ldots, y) \in I^L$$

and any positive integer n, it is easy to see that for the system (3), we have

$$F^n(\overrightarrow{x}) = (1-\varepsilon)(f^n(x), \quad f^n(x), \ldots, f^n(x))$$

and

$$F^n(\overrightarrow{y}) = (1-\varepsilon)(f^n(y), \quad f^n(y), \ldots, f^n(y)).$$

This implies that

$$F_{\overrightarrow{x}\overrightarrow{y}}((1-\varepsilon)\sqrt{L}t, F) = F_{xy}(t, f)$$

and

$$F^{\overrightarrow{x}\overrightarrow{y}}((1-\varepsilon)\sqrt{L}t, F) = F^{xy}(t, f)$$

for any $\overrightarrow{x}, \overrightarrow{y} \in I^L$. Write $D = diam(I^L)$. Since $|I| = 1$, we get that

$$\begin{aligned}
\mu_p(F) &\geq \sup_{x,y\in I} \frac{1}{D} \int_0^{+\infty} \left(F^*_{\overrightarrow{x}\overrightarrow{y}}(t, F) - F_{\overrightarrow{x}\overrightarrow{y}}(t, F)\right) dt \\
&= \sup_{x,y\in I} \frac{1}{\sqrt{L}} \int_0^{+\infty} \left(F^*_{\overrightarrow{x}\overrightarrow{y}}(t, F) - F_{\overrightarrow{x}\overrightarrow{y}}(t, F)\right) dt \\
&= \sup_{x,y\in I} \frac{1}{\sqrt{L}} \int_0^{+\infty} \left(F^*_{xy}(\frac{t}{(1-\varepsilon)\sqrt{L}}, f) - F_{xy}(\frac{t}{(1-\varepsilon)\sqrt{L}}, f)\right) dt \\
&= \sup_{x,y\in I} \int_0^{+\infty} \left(F^*_{xy}(t, f) - F_{xy}(t, f)\right) dt = (1-\varepsilon)\mu_p(f).
\end{aligned}$$

Thus, the proof is completed. □

Remark 3.1 It is known from [13] that the principal measure of system (1) is not less than $\frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}$ for any non-zero coupling constant $\varepsilon \in (0, 1)$ and the tent map Λ. However, by Eq. (1) in [13] and Theorem 3.2, the principal measure of system (3) is not less than $(1-\varepsilon)\left(\frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}\right)$ for any non-zero coupling constant $\varepsilon \in (0, 1)$ and the tent map Λ.

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